Published by Institute of Physics Publishing for SISSA

Received: *October 27, 2005*
Accepted: *November 7, 2005*
Published: *November 23, 2005*

The Bethe ansatz for $\mathbb{Z}_S$ orbifolds of $\mathcal{N}=4$ super Yang-Mills theory

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Abstract: Worldsheet techniques can be used to argue for the integrability of string theory on $AdS_5 \times S^5/\mathbb{Z}_S$, which is dual to the strongly coupled $\mathbb{Z}_S$-orbifold of $\mathcal{N}=4$ SYM. We analyze the integrability of these field theories in the perturbative regime and construct the relevant Bethe equations.

Keywords: AdS-CFT and dS-CFT Correspondence, Bethe Ansatz.

JHEP11(2005)037

Contents

1. Introduction

Our understanding of the AdS/CFT correspondence benefited greatly from the discovery of integrable structures both on the gauge theory [1 – 4] and string theory side [5 – 7] (see [8 – 11] for reviews). A natural question to ask is to what extent we can deform the model while preserving full integrability. This should lead to a better understanding of the integrable structures in large-N field theories in general and $\mathcal{N} = 4$ Super Yang-Mills (SYM) in particular.

Soon after the $\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$ correspondence was formulated [12 – 14], it was realized that modding out by a discrete subgroup of the R-symmetry group SU(4) leads to candidate dual pairs with reduced supersymmetry [15, 16]. The properties of the orbifold group determine the amount of preserved (super)symmetry. From the perspective of a four-dimensional theory, if the discrete orbifold group $\Gamma \in \mathrm{SU}(2)$, then one finds a $\mathcal{N} = 2$ superconformal gauge theory, while $\Gamma \in \mathrm{SU}(3)$ produces a $\mathcal{N} = 1$ superconformal gauge theory. For all other orbifold groups Γ the supersymmetry is completely broken. Depending on the precise embedding $\Gamma \subset \mathrm{SU}(4)$ some bosonic symmetries may survive the orbifold projection.

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Similar to the states of string theory orbifolds, the operators in orbifold field theories are organized in representations of the orbifold group. Evidence was provided in [17, 18] that all correlation functions of untwisted operators (i.e. the operators that do not transform under the quantum symmetry Γ) coincide in the planar limit with the correlation functions in the parent $\mathcal{N}=4$ SYM theory. Consequently, up to a trivial rescaling $\lambda \mapsto \lambda/|\Gamma|$, the anomalous dimensions of untwisted operators in the orbifold theory are the same as the parent $\mathcal{N}=4$ SYM theory. The anomalous dimensions of twisted operators do not obey such an inheritance principle and must be computed separately. As long as the orbifold quantum symmetry is unbroken[1] operators belonging to different twisted sectors do not mix under renormalization group flow and the anomalous dimensions of operators in a fixed twisted sector can be computed independently.

In this note we set up the Bethe ansatz for general abelian orbifolds, as a deformation of the Bethe ansatz for the parent theory. We begin by discussing in detail general $\mathbb{Z}_S$ orbifolds, explicitly computing the anomalous dimensions for several classes of operators. Examples of anomalous dimensions of such orbifolds of $\mathcal{N}=4$ SYM have already appeared in the literature in the context of plane wave orbifolds [23, 24] as well as in connection with the integrability of the dilatation operator in particular sectors of the theory [25, 26]. We will compare them with the proposed Bethe ansatz. We then extend these results to the general case. We close with a discussion on further deformations of orbifolds of $\mathcal{N}=4$ SYM as well as orbifolds of other deformations of $\mathcal{N}=4$ SYM.

2. Orbifolds of field theory

We start by reviewing the construction of orbifolds for gauge field theories and set up our notation. For these models we find the action of the planar dilatation generator and apply it to find a few sets of anomalous dimensions.

2.1 Fields

We consider field theories dual to superstrings on $AdS_5 \times (S^5/\Gamma)$ where Γ is a discrete subgroup of SU(4). In the gauge theory dual, the orbifold group Γ acts on the gauge group as well as on the SU(4) R-symmetry group. Consistency of stringy construction of such field theories requires that we pick regular representations of the orbifold group [17]. The field X transforms under the action of an element of Γ as follows

$$X \mapsto \gamma(R_\gamma X)\gamma^{-1} = X\,. \tag{2.1}$$

Here γ is the representation of the element as a matrix of the gauge group and R_γ is the representation in SU(4). The orbifold group acts trivially and therefore (2.1) is a constraint for the fields X.

[1]Non-supersymmetric orbifold theories exhibit certain instabilities [19, 20] which may be interpreted in terms of the condensation of closed string tachyons [21, 22]. In the condensation process the quantum symmetry of the orbifold is spontaneously broken potentially leading to a more complicated structure of the dilatation operator.

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In particular, we are interested in a $\Gamma = \mathbb{Z}_S$ orbifold of $\mathcal{N} = 4$ SYM with gauge group $\mathrm{U}(SN)$. The orbifold theory has residual $\mathrm{U}(N)^S$ local symmetry. We introduce a $SN \times SN$ matrix γ representing a generator of $\mathbb{Z}_S$. Using $N \times N$ blocks it has the form

$$\begin{aligned}\gamma &= \mathrm{diag}(1, e^{2\pi i/S}, e^{4\pi i/S}, \dots, e^{-2\pi i/S}) \\ &= \mathrm{diag}(1, \omega, \omega^2, \dots, \omega^{S-1}), \qquad \omega = e^{2\pi i/S}\,.\end{aligned} \tag{2.2}$$

A field X of the orbifold theory with definite charges of $\mathfrak{su}(4)$ is defined by the constraint

$$X = \exp\left(\frac{2\pi i s_X}{S}\right) \gamma X \gamma^{-1} = \omega^{s_X}\, \gamma X \gamma^{-1}\,. \tag{2.3}$$

Alternatively, we can project the $\mathcal{N} = 4$ SYM field $X^{\mathcal{N}=4}$ to the orbifold field X by means of

$$X = \frac{1}{S} \sum_{k=1}^{S} \exp\left(\frac{2\pi i k s_X}{S}\right) \gamma^k X^{\mathcal{N}=4} \gamma^{-k}\,. \tag{2.4}$$

The $\mathbb{Z}_S$ integer s_X selects which of the (secondary) diagonals of the $SN \times SN$ matrix X is occupied. The phase s_X is coupled to the internal $\mathfrak{su}(4)$ symmetry as follows

$$s_X = \mathbf{t} \cdot \mathbf{q}_X\,. \tag{2.5}$$

The vector $\mathbf{q}_X$ represents the $\mathfrak{su}(4)$ charges of the field X and the vector $\mathbf{t}$ contains the parameters of the orbifold. As $\mathfrak{su}(4)$ has rank three, there are three independent parameters t_1, t_2, t_3. These are taken to be integers and only their remainder modulo S is relevant. Let us define s_X through the action on a spinor of $\mathfrak{so}(6)$. The phases for the four components of the spinor shall be given by

$$\mathbf{s_4} = (-t_1, t_1 - t_2, t_2 - t_3, t_3)\,. \tag{2.6}$$

Vectors of $\mathfrak{so}(6)$ are given by bi-spinors and conjugate spinors by triple spinors; their phases $\mathbf{s_6}$ and $\mathbf{s_{\bar{4}}}$ can therefore be obtained by adding the corresponding phases from $\mathbf{s_4}$. Fields invariant under $\mathfrak{su}(4)$ have no phases and consequently they are block-diagonal, such as the gauge field $\mathcal{A}_\mu$. For $\mathcal{N} = 1$ supersymmetric theories, it is useful to split up $\mathbf{6} = \mathbf{3} + \mathbf{3}$ and employ a $\mathfrak{su}(3)$ notation for the fields. Then the triplet of complex scalars is related to the scalar as a bi-spinor as $\phi_k = \Phi_{k4}$ and the corresponding weights are

$$\mathbf{s_3} = (-t_1 + t_3, t_1 - t_2 + t_3, t_2)\,. \tag{2.7}$$

2.2 Spin chains

A generic local operator invariant under $\mathrm{U}(N)^S$ is given by traces of products of the fields $\mathcal{W}$ and the $\mathbb{Z}_S$ generator γ. Due to the orbifold identity (2.3), which can be reformulated as

$$X\gamma = \exp\left(\frac{2\pi i s_X}{S}\right) \gamma X\,, \tag{2.8}$$

it is possible to collect all $\mathbb{Z}_S$ generators at one point within a trace. Single-trace operators are therefore spanned by

$$\mathcal{O} = \mathrm{Tr}\, \gamma^T \mathcal{W}_{A_1} \mathcal{W}_{A_2} \cdots \mathcal{W}_{A_L}\,. \tag{2.9}$$

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Here $\mathcal{W} \in \{\mathcal{D}^n\Phi, \mathcal{D}^n\Psi, \mathcal{D}^n\mathcal{F}\}$ is a multiple derivative of one of the fields of the theory. Some of the operators (2.9) vanish identically. Using the equation (2.8) to commute one γ past all the fields in the operator leads to the conclusion that the necessary and sufficient condition for non-vanishing operators is that their total $\mathbb{Z}_S$ charge is zero, i.e.:

$$\frac{1}{S}\sum_{k=1}^{L} s_{A_k} \in \mathbb{Z}\,. \tag{2.10}$$

Up to the cyclicity of the trace, this local operator is isomorphic to some spin chain state

$$|\mathcal{O}\rangle = |\gamma^T; A_1, A_2, \ldots, A_L\rangle\,. \tag{2.11}$$

In this picture, the gauge theory generator of anomalous dimensions maps to the spin chain hamiltonian $\mathcal{H}$. The energy eigenvalues E of $\mathcal{H}$ are related to the anomalous dimensions of local operators by

$$\delta D_{\mathcal{O}} = \frac{g_{\mathrm{YM}}^2 N}{8\pi 2}\, E_{|\mathcal{O}\rangle}\,. \tag{2.12}$$

By investigating Feynman diagrams, it can be seen that the L-loop hamiltonian commutes with the orbifolding procedure in the large-N limit for all operators of length strictly larger than L [18].[2,3] This means that the hamiltonian for $\mathcal{N} = 4$ SYM is the same as for the orbifolded theory when there are no $\mathbb{Z}_S$ generators involved. For example, if the nearest-neighbor hamiltonian for $\mathcal{N} = 4$ SYM at the leading, one-loop level is given by

$$\mathcal{H}^{\mathcal{N}=4}_{[12]}\, \mathcal{W}^{\mathcal{N}=4}_A \mathcal{W}^{\mathcal{N}=4}_B = \mathcal{H}^{CD}_{AB}\, \mathcal{W}^{\mathcal{N}=4}_C \mathcal{W}^{\mathcal{N}=4}_D\,. \tag{2.13}$$

Then the action of the orbifolded hamiltonian yields

$$\mathcal{H}_{[12]}\, \mathcal{W}_A \mathcal{W}_B = \mathcal{H}^{CD}_{AB}\, \mathcal{W}_C \mathcal{W}_D\,. \tag{2.14}$$

If $\mathbb{Z}_S$ generators are present in the interaction region they should first be shifted away using equation (2.8), e.g.

$$\mathcal{H}_{[12]}\, \mathcal{W}_A \gamma^k \mathcal{W}_B = \exp\left(\frac{2\pi i k s_A}{S}\right) \mathcal{H}_{[12]}\, \gamma^k \mathcal{W}_A \mathcal{W}_B = \exp\left(\frac{2\pi i k s_A}{S}\right) \mathcal{H}^{CD}_{AB}\, \gamma^k \mathcal{W}_C \mathcal{W}_D\,. \tag{2.15}$$

Clearly, the number of $\mathbb{Z}_S$ generators, i.e. the $\mathbb{Z}_S$ charge T of the local operator is preserved by $\mathcal{H}$. Therefore the states (2.11) with fixed T form a so-called T-twisted sector of the model.

[2] A further subtlety arises for non-supersymmetric orbifold actions, as quantum corrections require the modification of the tree-level action by certain bilinears in twisted dimension two operators. The planar anomalous dimensions of length L operators cannot be not affected by such deformations before $(L-1)$ loops. In the following we will largely ignore this subtlety.

[3] It is not immediately clear whether the orbifold procedure commutes with the L-loop spin chain hamiltonian for operators of length smaller than L. It is intriguing that this apparent problem sets in at the same step at which the wrapping interactions start being relevant [27] (see also [28] for recent investigations of this effect). It would be interesting to establish whether a connection exists between these two effects and if their mutual consistency constrains the form of the latter.

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2.3 Anomalous dimensions

It is straightforward to use the description above to compute the anomalous dimensions of one-excitation states

$$\mathcal{O} = \mathrm{Tr}\left(\gamma^T \mathcal{W} \mathcal{Z}^{L-1}\right), \tag{2.16}$$

where $\mathcal{Z}$ is one of the complex scalars and $\mathcal{W}$ can be any of the fundamental fields of the theory except for the gauge field. The orbifold phases (weights) of the fields $\mathcal{Z}, \mathcal{W}$ are assumed to be $s_{\mathcal{Z}}, s_{\mathcal{W}}$, respectively. In general, for an operator to be non-trivial it is necessary to accumulate a trivial phase as γ is moved past all the fields using (2.8). In the case at hand (2.10) reduces to

$$\frac{(L-1)s_{\mathcal{Z}} + s_{\mathcal{W}}}{S} \in \mathbb{Z}\,. \tag{2.17}$$

For those states, the energy E, alias the one-loop planar anomalous dimension $\delta D = (g_{\mathrm{YM}}^2 N/8\pi^2)E$, is

$$E = 4\sin^2\frac{\pi s_{\mathcal{Z}} T}{S}\,. \tag{2.18}$$

Given the simplicity of the states of length two, it is also find relatively closed forms for the anomalous dimensions of all operators of this length descending from the $\mathfrak{so}(6)$ sector. Only the operators $\mathrm{Tr}(\gamma^T\{\bar\phi_i, \phi^i\})$ exist for a general abelian orbifold group. Their anomalous dimensions are given by the eigenvalues of the mixing matrix

$$E = \begin{pmatrix} 2+2\sin^2(\pi s_1 T/S) & 2-2\sin^2(\pi s_1 T/S) & 2-2\sin^2(\pi s_1 T/S) \\ 2-2\sin^2(\pi s_2 T/S) & 2+2\sin^2(\pi s_2 T/S) & 2-2\sin^2(\pi s_2 T/S) \\ 2-2\sin^2(\pi s_3 T/S) & 2-2\sin^2(\pi s_3 T/S) & 2+2\sin^2(\pi s_3 T/S) \end{pmatrix}, \tag{2.19}$$

which are relatively complicated functions of the orbifold weights $\mathbf{s_3} = (s_1, s_2, s_3)$ in the $\mathbf{3}$ representation. If two weights are equal, $\mathbf{s_3} = (2s', s, s)$, the eigenvalues take simple forms:

$$\begin{aligned} E_1 &= 4\sin^2\left(\pi s\frac{T}{S}\right), \qquad (2.20)\\ E_{2,3} &= 3+\sin^2\left(2\pi s'\frac{T}{S}\right) \pm \cos\left(2\pi s'\frac{T}{S}\right)\sqrt{9-8\sin^2\left(\pi s\frac{T}{S}\right)+\sin^2\left(2\pi s'\frac{T}{S}\right)}. \end{aligned}$$

This is because the orbifold preserves a $\mathfrak{su}(2)$ symmetry (acting on the scalars ϕ_2 and ϕ_3) which relates one of the three states with the single-excitation state implying that E_1 must coincide with (2.18). Note that the only untwisted operator with two fields and non-trivial anomalous dimension is an orbifold descendant of the Konishi operator and inherits its anomalous dimension $E = 6$. Other choices of orbifold weights can also be analyzed, but they lead to lengthy expressions for the anomalous dimensions which we will not list here.

3. Bethe ansatz

From the standpoint of the string theory dual it is easy to argue that, in the strong 't Hooft coupling limit, orbifold field theories of $\mathcal{N} = 4$ SYM are integrable. Assuming integrability, we shall investigate the gauge theory Bethe ansatz for the orbifold model. Let us first of all review integrability for the parent $\mathcal{N}=4$ supersymmetric model. An eigenstate on a spin chain is determined by a set of excitations. There are eight types of excitations for the model, labeled by $j=0,\ldots,7$. Excitations $j=1,\ldots,7$ correspond to spin waves which change the flavors of spin sites. They are associated with a spectral parameter $u_{j,k}$ where $k=1,\ldots K_j$ enumerates the various excitations of type j. The quasi-excitation $j=0$ corresponds to the insertion of a new spin chain site. This type of excitation does not have an associated spectral parameter, so it suffices to give the number $K_0=L$, which represents the length of the spin chain.

3.1 Equations for $\mathcal{N}=4$ SYM

The leading order (one-loop) Bethe equations for $\mathcal{N}=4$ SYM read [3]

$$\left(\frac{u_{j,k}-\frac{i}{2}V_j}{u_{j,k}+\frac{i}{2}V_j}\right)^L \prod_{\substack{j'=1\\(j',k')\neq(j,k)}}^{J}\prod_{k'=1}^{K_{j'}} \frac{u_{j,k}-u_{j',k'}+\frac{i}{2}M_{j,j'}}{u_{j,k}-u_{j',k'}-\frac{i}{2}M_{j,j'}}=1\,, \qquad \prod_{j=1}^{J}\prod_{k=1}^{K_j}\frac{u_{j,k}+\frac{i}{2}V_j}{u_{j,k}-\frac{i}{2}V_j}=1\,, \tag{3.1}$$

where $J=7$ is the rank of $\mathfrak{su}(2,2|4)$, $M_{j,j'}$ is the symmetric Cartan matrix and V_j are the labels which specify the representation of spin sites. The first equation is the Bethe equation for the excitation k of type j. It ensures that the spin chain state is periodic. The second equation is the momentum constraint which ensures the cyclicity of the state so that we can interpret it as a single-trace local operator. It is useful to view the momentum constraint as the Bethe equation for the quasi-excitations of type 0: Periodicity for spin sites is equivalent to vanishing momentum. Although there are L quasi-excitations of type 0, there is only one corresponding Bethe equation, because all of these quasi-excitations are equivalent, they have no spectral parameter which might distinguish them. We can therefore summarize all the Bethe equations as

$$\prod_{\substack{j'=0\\(j',k')\neq(j,k)}}^{J}\prod_{k'=1}^{K_{j'}} S_{j,j'}(u_{j,k},u_{j',k'})=1 \tag{3.2}$$

with the scattering phases $S_{j,j'}=1/S_{j',j}$ of the excitations

$$S_{j,j'}=\frac{u_{j,k}-u_{j',k'}+\frac{i}{2}M_{j,j'}}{u_{j,k}-u_{j',k'}-\frac{i}{2}M_{j,j'}}\,,\qquad S_{j,0}=\frac{1}{S_{0,j}}=\frac{u_{j,k}-\frac{i}{2}V_j}{u_{j,k}+\frac{i}{2}V_j}\,,\qquad S_{0,0}=1\,. \tag{3.3}$$

The energy for a periodic eigenstate, alias the planar anomalous dimension of a local operator, is given by

$$E=\sum_{j=0}^{J}\sum_{k=1}^{K_j} e_j(u_{j,k})\,,\qquad e_j(u_{j,k})=\frac{i}{u_{j,k}+\frac{i}{2}V_j}-\frac{i}{u_{j,k}-\frac{i}{2}V_j}\,,\qquad e_0=0\,. \tag{3.4}$$

3.2 Orbifolding the Bethe ansatz

Keeping the above discussion in mind, the constraint (2.8) suggests how to extend the Bethe ansatz to orbifolds: This equation determines the phase shift $2\pi s_X/S$ for exchanging a flavored spin chain site X with the $\mathbb{Z}_S$ generator γ. Therefore it is straightforward to

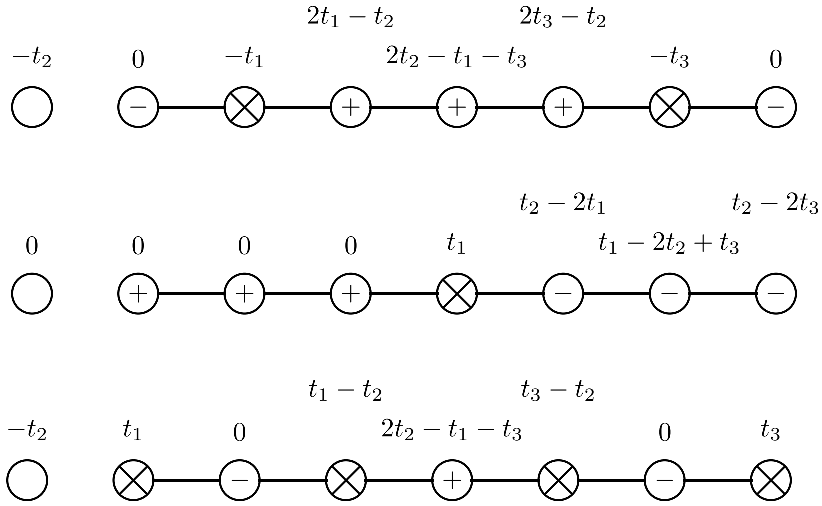

Figure 1: Orbifold weights for the simple roots using the Dynkin diagrams "Beauty", "Beast" and "Higher" (from top to bottom). The leftmost root represents a site of the spin chain, a quasi-excitation of type 0. The indicated numbers are the orbifold weights s_j for each type of Bethe root.

represent γ by a new type of quasi-excitation, i.e. we introduce excitations of type $j = -1$ and set the lower bound for j' in (3.2) to $j' = -1$. Like the quasi-excitation with $j = 0$ it does not have an associated spectral parameter. The only relevant parameter is $K_{-1} = T$ which specifies the twisted sector. The phase shift

$$S_{j,-1} = \frac{1}{S_{-1,j}} = \exp\left(2\pi i \frac{s_j}{S}\right) \tag{3.5}$$

for exchanging $j' = -1$ with any of the other $j = 0, \ldots, 7$ has to be adjusted so that (2.8) is respected. This is simplified by the fact that the phase shifts are determined by (2.5) through the $\mathfrak{su}(4)$ charges. Consequently, the numbers s_j are defined by (2.5) through the charges $\mathbf{q}_j$ of simple roots of the algebra. For superalgebras there are several equivalent choices of simple roots, and thus Dynkin diagrams and Cartan matrices $M_{j,j'}$. There are three useful choices for $\mathfrak{su}(2,2|4)$ which we label by "Beauty", "Beast" and "Higher", see figure 1. The numbers s_j, $j = (0|\,1, \ldots, 7)$, for those three choices are given by

$$\begin{aligned}
\mathbf{s}_{\text{Beauty}} &= (-t_2|\,0, -t_1, 2t_1 - t_2, 2t_2 - t_1 - t_3, 2t_3 - t_2, -t_3, 0)\,,\\
\mathbf{s}_{\text{Beast}} &= (0|\,0, 0, 0, t_1, t_2 - 2t_1, t_1 - 2t_2 + t_3, t_2 - 2t_3)\,,\\
\mathbf{s}_{\text{Higher}} &= (-t_2|\,t_1, 0, t_1 - t_2, 2t_2 - t_1 - t_3, t_3 - t_2, 0, t_3)\,.
\end{aligned} \tag{3.6}$$

Finally, the scattering phase of two γ's is trivial because they commute. They also do not contribute to the energy directly

$$S_{-1,-1} = 1\,, \qquad e_{-1} = 0\,. \tag{3.7}$$

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The one-loop Bethe equations for a $\mathbb{Z}_S$ orbifold of $\mathcal{N}=4$ SYM are just as in (3.2)

$$\prod_{\substack{j'=-1\\(j',k')\neq(j,k)}}^{J}\prod_{k'=1}^{K_{j'}} S_{j,j'}(u_{j,k},u_{j',k'})=1\,,\qquad E=\sum_{j=-1}^{J}\sum_{k=1}^{K_j}e_j(u_{j,k}) \tag{3.8}$$

but extended by the quasi-excitations of type $j,j'=-1$. Written out explicitly, the Bethe equations read

$$e^{2\pi iTs_j/S}\left(\frac{u_{j,k}-\frac{i}{2}V_j}{u_{j,k}+\frac{i}{2}V_j}\right)^L\prod_{\substack{j'=1\\(j',k')\neq(j,k)}}^{J}\prod_{k'=1}^{K_{j'}}\frac{u_{j,k}-u_{j',k'}+\frac{i}{2}M_{j,j'}}{u_{j,k}-u_{j',k'}-\frac{i}{2}M_{j,j'}}=1\,. \tag{3.9}$$

Furthermore, there is the momentum constraint for $j=0$

$$e^{2\pi iTs_0/S}\prod_{j'=1}^{J}\prod_{k'=1}^{K_{j'}}\frac{u_{j',k'}+\frac{i}{2}V_{j'}}{u_{j',k'}-\frac{i}{2}V_{j'}}=1 \tag{3.10}$$

and the twist constraint for $j=-1$

$$e^{-2\pi iLs_0/S}\prod_{j'=1}^{J}e^{-2\pi iK_{j'}s_{j'}/S}=1\,. \tag{3.11}$$

These equations agree with the special cases studied in [25, 26].

The Bethe vacuum leading to these equations is $\operatorname{Tr}\gamma^T\mathcal{Z}^J$. However, depending on the peculiarities of the orbifold group and its chosen action, the twist constraint may project out the Bethe vacuum while keeping some of the excited states. This is somewhat different from the string theory orbifold construction in which, at least in a Green-Schwarz formulation, the ground state is always part of the spectrum. Since we assumed that the twisted sectors are decoupled, it may be possible to construct one-loop Bethe equations whose form explicitly depends on the twisted sector and length of the operators, such that the Bethe vacuum is not projected out. However, the explicit dependence on the length prevents a direct generalization to higher loops. As we will see shortly in section 3.5, the setup outlined here is compatible with the higher-loop Bethe ansatz for $\mathcal{N}=4$ SYM and quantum strings [29].

Note that the modification of the Bethe equations is reminiscent of fractional magnetic flux surrounded by the spin chain. In this picture, there are T units of $1/S$ fractional flux associated to the T-twisted sector. The Bethe roots have electric charges s_j. This means the particles undergo an Aharonov-Bohm phase shift of $2\pi s_jT/S$ when moving once around the spin chain as in eqs. (3.9), (3.10). The twist constraint (3.10) represents the Dirac charge quantisation condition. The twist matrix γ as a quasi-excitation can be viewed as the boundary of a coordinate patch similar to Dirac strings or branch cuts. It does not carry a momentum and moving it around corresponds to changing coordinate patches by a gauge transformation.

3.3 Energies

Using these equations it is rather straightforward to recover the one-loop anomalous dimensions computed before as well as some results existing in the literature [24, 26, 25]. Quite obviously, the simplest states to analyze are those with a single excitation of type $j = 4$ above the Bethe vacuum. Of course, such operators do not exist in all theories, as they must satisfy the twist constraint (3.11). This is the equation (2.10) expressed in terms of Bethe roots and for a single excitation it is equivalent to (2.17). When this constraint is satisfied, the (unique) rapidity is determined by the momentum constraint (3.10) as

$$u = -\frac{1}{2} \cot \frac{\pi s_0 T}{S} \,. \tag{3.12}$$

Substituting this in (3.4) reproduces the anomalous dimension (2.18) obtained from the explicit application of the dilatation operator.

Our second test concerns the operators whose anomalous dimensions are listed in (2.21). They correspond to length-two states of the spin chain with two main excitations; for the "Beauty" Dynkin diagram, the number of excitations for each node is

$$(L|\, K_1, K_2, K_3, K_4, K_5, K_6, K_7) = (2|\, 0, 0, 1, 2, 1, 0, 0) \,. \tag{3.13}$$

According to (2.7) the choice $\mathbf{s_3} = (2s', s, s)$ is related $\mathbf{t} = (s - s', s, s + s')$ which again implies the weights $\mathbf{s}_{\text{Beauty}} = (-s|\, 0, -s{+}s', s{-}2s', 0, s{+}2s', -s{-}s', 0)$ for Bethe roots. The equations (3.9) can be solved explicitly as solutions to a bi-quadratic equation. Evaluating (3.4) on these solutions leads to the anomalous dimensions (2.21).

Last, but not least, it is easy to recover the anomalous dimensions of BMN-type operators with excitations of a single type, originally discussed in [24, 26, 25]. In this case only the orbifold weight corresponding to $j = 4$ (in the "Beauty" form) is relevant. Then, assuming that the number of excitations is such that the operator twist constraint is satisfied, the logarithm of the Bethe equations implies that

$$u_{4,k} = \frac{L}{2\pi(n_k + T s_4/S)} \qquad \text{with} \qquad \sum_{k=1}^{K} n_k = 0 \,. \tag{3.14}$$

With these rapidities, the equation (3.4) yields the known result

$$E = \frac{4\pi^2}{L^2} \sum_{k=1}^{K} \left(n_k + \frac{T s_4}{S} \right)^2 \,. \tag{3.15}$$

3.4 Specific orbifolds

Let us investigate the orbifold parameters for several special cases. The first is an orbifold preserving $\mathcal{N} = 1$ superconformal symmetry. This is easily achieved by demanding that in the "Beast" representation (3.6), the fermionic excitation $j = 4$ commutes with the $\mathbb{Z}_S$ generator $j' = -1$. In other words, we should not not only have unbroken $\mathfrak{su}(2,2)$ ($s_1 = s_2 = s_3 = 0$ is guaranteed) but also unbroken $\mathfrak{su}(2,2|1)$, i.e. $s_4 = 0$. This means we set $t_1 = 0$ and obtain for a generic $\mathcal{N} = 1$ orbifold

$$\begin{aligned}
\mathbf{s_4} &= (0, -t_2, t_2 - t_3, t_3)\,, \qquad \mathbf{s_3} = (t_3, t_3 - t_2, t_2)\,, \\
\mathbf{s}_{\text{Beauty}} &= (-t_2 |\, 0, 0, -t_2, 2t_2 - t_3, 2t_3 - t_2, -t_3, 0)\,, \\
\mathbf{s}_{\text{Beast}} &= (0 |\, 0, 0, 0, 0, t_2, t_3 - 2t_2, t_2 - 2t_3)\,, \\
\mathbf{s}_{\text{Higher}} &= (-t_2 |\, 0, 0, -t_2, 2t_2 - t_3, t_3 - t_2, 0, t_3)\,.
\end{aligned} \tag{3.16}$$

To go to a $\mathcal{N} = 2$ orbifold we need to have $s_4 = s_5 = 0$ in the Beast representation, i.e. $t_1 = t_2 = 0$. Without loss of generality we can then set $t_3 = 1$, i.e. $\mathbf{t} = (0, 0, 1)$, and obtain

$$\begin{aligned}
\mathbf{s_4} &= (\ \ 0, \ \ 0, -1, +1)\,, \qquad \mathbf{s_3} = (+1, +1, \ \ 0)\,, \\
\mathbf{s}_{\text{Beauty}} &= (\ \ 0 | \ \ 0, \ \ 0, \ \ 0, -1, +2, -1, \ \ 0)\,, \\
\mathbf{s}_{\text{Beast}} &= (\ \ 0 | \ \ 0, \ \ 0, \ \ 0, \ \ 0, \ \ 0, +1, -2)\,, \\
\mathbf{s}_{\text{Higher}} &= (\ \ 0 | \ \ 0, \ \ 0, \ \ 0, -1, +1, \ \ 0, +1)\,.
\end{aligned} \tag{3.17}$$

For $S = 2$ the symmetry is enhanced by $\mathfrak{su}(2)$.

We could also demand a preserved $\mathfrak{su}(3)$ symmetry. This is achieved for $s_5 = s_6 = 0$ or $6t_1 = 3t_2 = 2t_3$. Without loss of generality we set $\mathbf{t} = (1, 2, 3)$ and obtain

$$\begin{aligned}
\mathbf{s_4} &= (-1, -1, -1, +3)\,, \qquad \mathbf{s_3} = (+2, +2, +2)\,, \\
\mathbf{s}_{\text{Beauty}} &= (-2 | \ \ 0, -1, \ \ 0, \ \ 0, +4, -3, \ \ 0)\,, \\
\mathbf{s}_{\text{Beast}} &= (\ \ 0 | \ \ 0, \ \ 0, \ \ 0, +1, \ \ 0, \ \ 0, -4)\,, \\
\mathbf{s}_{\text{Higher}} &= (-2 | +1, \ \ 0, -1, \ \ 0, +1, \ \ 0, +3)\,.
\end{aligned} \tag{3.18}$$

The cases $S = 2, 4$ are special, they are the only orbifolds with preserved $\mathfrak{su}(4)$ symmetry. For $S = 3$ we have $\mathcal{N} = 1$ superconformal symmetry instead.

Another interesting option is to preserve $\mathfrak{su}(2) \times \mathfrak{su}(2)$ symmetry. This is achieved for $s_5 = s_7 = 0$ or $t_2 = 2t_3 = 2t_1$. Without loss of generality we set $\mathbf{t} = (1, 2, 1)$ and the weights in various representations read

$$\begin{aligned}
\mathbf{s_4} &= (-1, -1, +1, +1)\,, \qquad \mathbf{s_3} = (\ \ 0, \ \ 0, +2)\,, \\
\mathbf{s}_{\text{Beauty}} &= (-2 | \ \ 0, -1, \ \ 0, +2, \ \ 0, -1, \ \ 0)\,, \\
\mathbf{s}_{\text{Beast}} &= (\ \ 0 | \ \ 0, \ \ 0, \ \ 0, +1, \ \ 0, -2, \ \ 0)\,, \\
\mathbf{s}_{\text{Higher}} &= (-2 | +1, \ \ 0, -1, +2, -1, \ \ 0, +1)\,.
\end{aligned} \tag{3.19}$$

3.5 Higher loops, quantum strings

We would now like to see whether the orbifold is compatible with the Bethe ansatz for higher-loop $\mathcal{N} = 4$ SYM [29]. These equations seem to be equally well applicable to quantum strings on $AdS_5 \times S^5$ when introducing an additional overall phase for the strong coupling regime [30]. In these ansätze one uses the rapidities $x_{j,k}$ instead of $u_{j,k}$ and the scattering phases $S_{j,j'}(x_{j,k}, x_{j',k'})$ between the various kinds of Bethe roots are modified to accommodate the higher-loop interactions, cf. [29] for full expressions. The generic form of the Bethe equations, however, remains unchanged as compared to (3.2). In these Bethe equations we cannot use any Dynkin diagram, but we should restrict to the one we denoted by "Higher" in figure 1. Another very important ingredient for the consistency of the equations is the "dynamic transformation" [29] which ensures compatibility with the dynamic spin chains that arise at higher loops [4]. This transformation is the result of a symmetry of the scattering phases

$$S_{j,3}(x,x_3) = S_{j,1}(x,x_1)\,S_{j,0}(x)\,, \qquad S_{j,5}(x,x_5) = S_{j,7}(x,x_7)\,S_{j,0}(x) \tag{3.20}$$

when $x_1 x_3 = x_5 x_7 = g_{\mathrm{YM}}^2 N/16\pi^2$. The scattering phases for $\mathcal{N}=4$ SYM with $j = 0,\ldots 7$ obey this symmetry. In order for the orbifold to be compatible with the transformation, the equation (3.20) should hold for $j=-1$ as well:

$$e^{2\pi i s_3/S} = e^{2\pi i s_1/S} e^{2\pi i s_0/S}\,, \qquad e^{2\pi i s_5/S} = e^{2\pi i s_7/S} e^{2\pi i s_0/S}\,. \tag{3.21}$$

This means that $s_1 + s_0 = s_3$ and $s_5 = s_7 + s_0$ in the "Higher" choice of Dynkin diagram, which is indeed satisfied by (3.6). Therefore the higher-loop Bethe equations are consistent with the orbifolding procedure and (3.8) will most likely produce equally accurate results for orbifolds as the original equations (3.2) for plain $\mathcal{N}=4$ SYM.[4]

4. Generalizations

In general, the number of deformations of a field theory is quite large. Deformations preserving certain properties of the parent theory — such as integrability for example — are less numerous. If a theory exhibits at least one global U(1) symmetry it is always possible to construct its abelian orbifold descendants which, following the arguments described above, exhibit an integrable dilatation operator. If the parent theory has a string theory dual, a similar conclusion can be reached in the strong coupling regime.

Our discussion naturally generalizes to arbitrary abelian orbifolds. Since all group generators commute, they can be simultaneously diagonalized and each of them spans a copy of $\mathbb{Z}_S$ for some S. Thus, a general abelian orbifold is isomorphic to an orbifold by a product of $\mathbb{Z}_{S_j}$ factors, $\otimes_{j=1}^{P}\mathbb{Z}_{S_j}$, and independent orbifold weights $\mathbf{t}_j$ for each factor. The operators are organized in (twisted) sectors with respect to each factor of the orbifold group, which are labelled by T_j. Then starting from eqs. (3.9), (3.10), the relevant Bethe equations are then obtained via the following replacements: $S \to S_{j'}$, $T \to T_{j'}$, $s_{j'} \to s_{j',j}$. The index $j' = 1,\ldots,P$ runs over all the factors in the orbifold group. The equation (3.11) is replaced by a system of equations enforcing the fact that the eigenoperators are built out of fields invariant under each factor of the orbifold group.

Another direction for generalizations are so-called β-deformations. If a theory exhibits several commuting U(1) symmetries it is possible to construct its β-deformation [31–35].

[4] As we have mentioned before, due to the quasi-excitation representing the orbifold group element, the spin chain hamiltonian for finite-length operators potentially differs from the one of $\mathcal{N}=4$ SYM one loop order before the effects of wrapping interactions become relevant. It is therefore possible that the prediction of the Bethe ansatz for the anomalous dimensions of length L operators depart from their actual anomalous dimensions at $(L-1)$ loops.

It was argued in [34, 36] that this deformation preserves integrability in the weak coupling regime. If the parent theory has a string theory dual, then the dual of the deformed theory can also be constructed [33] and it can be argued that integrability is preserved also in the strong coupling regime [34]. From this description it is clear that it is possible to combine these two deformations. Moreover, as both deformations preserve the existing U(1) symmetries, the order in which the deformations are performed is inconsequential. If the parent theory is $\mathcal{N} = 4$ SYM, the Bethe equations eqs. (3.9), (3.10), (3.11) are modified by adjoining to the left hand side the β-dependent phase deformations discussed in [36, section 4]. In short, an antisymmetric matrix $A_{j,j'}$ is introduced which gives an additional constant phase for commuting one Bethe root of type j past a Bethe root of type j'.

We now assemble the above two generalizations with the higher-loop equations of section 3.5: The generic higher-loop equations for deformed, multiply orbifolded $\mathcal{N} = 4$ SYM read ($j = 1, \ldots, 7$, $k = 1, \ldots, K_j$)

$$\prod_{j'=1}^{P} e^{2\pi i T_{j'} s_{j',j}/S_{j'}}\, e^{iA_{j,0}L} \prod_{j'=1}^{7} e^{iA_{j,j'}K_{j'}}\, S_{j,0}(x_{j,k})^L \prod_{\substack{j'=1\\(j',k')\neq(j,k)}}^{7} \prod_{k'=1}^{K_{j'}} S_{j,j'}(x_{j,k}, x_{j',k'}) = 1\,. \tag{4.1}$$

Furthermore, there is the momentum constraint

$$\prod_{j'=1}^{P} e^{2\pi i T_{j'} s_{j',0}/S_{j'}} \prod_{j'=1}^{7} e^{iA_{0,j'}K_{j'}} \prod_{j'=1}^{7} \prod_{k'=1}^{K_{j'}} S_{0,j'}(x_{j',k'}) = 1\,, \tag{4.2}$$

and the twist constraint ($j = 1, \ldots, P$)

$$e^{-2\pi i s_{j,0}L/S_j} \prod_{j'=1}^{7} e^{-2\pi i s_{j,j'}K_{j'}/S_j} = 1\,. \tag{4.3}$$

Consistency of the dynamic transformation (3.20) leads to the following restrictions

$$\begin{aligned} A_{j,0} + A_{j,1} &= A_{j,3}\,, \qquad & s_{j,0} + s_{j,1} &= s_{j,3}\,, \\ A_{j,0} + A_{j,7} &= A_{j,5}\,, \qquad & s_{j,0} + s_{j,7} &= s_{j,5}\,. \end{aligned} \tag{4.4}$$

Note that this constraint is even more general than the section 3.2: It also includes the possibility to orbifold and deform parts of the AdS_5 space.

When assigning the P twist matrices γ_j as quasi-excitations $j = -1, \ldots, -P$ we can also write the Bethe equations in short as

$$\prod_{\substack{j'=-P\\(j',k')\neq(j,k)}}^{7} \prod_{k'=1}^{K_{j'}} S^A_{j,j'}(x_{j,k}, x_{j',k'}) = 1\,, \tag{4.5}$$

with suitably twisted scattering phases $S^A_{j,j'}$ for between all types of excitations $j, j' = 1, \ldots, 7$, spin chain sites $j, j' = 0$ and twist matrices $j, j' = -1, \ldots, -P$.

It is worth emphasizing that, while both the β-deformation and the orbifold projection lead to similar modifications to the Bethe equations, they are in fact quite different. In physical terms, the difference is similar to that between assigning (non-commuting) magnetic charges to all spin states vs. having a system with fractional magnetic flux (the orbifold field theory), cf. section 3.2. If the number of excitations is small compared to the length of the chain (the BMN limit) the two physical pictures become similar, as the large number of fields constituting the BMN vacuum may be interpreted as a twist matrix provided its charge is identified with the orbifold twist. This is reflected by the similarity of the anomalous dimensions in this limit, which are given by the original BMN spectrum with mode numbers shifted by a fixed fractional amount.

The quasi-excitation representing the orbifold group element has a physical interpretation closely related to string theory. On the orbifold covering space, twisted strings are interpreted as open strings in which the world sheet fields the two ends are related by the action of the orbifold group element. Similarly here, it is possible to interpret the spin chain for the orbifold theory as an open spin chain with boundary conditions specified by an orbifold group element.[5] The quasi-excitation representing the orbifold group element is an effective way of implementing this boundary condition while keeping the spin chain closed, similarly with the twisted boundary conditions for closed strings on orbifolds.

5. Discussions and outlook

In this paper we have discussed in detail the integrability of orbifolds of four-dimensional field theories with integrable dilatation operators, with emphasis on $\mathcal{N} = 4$ SYM. We showed how to modify the Bethe equations to incorporate an abelian orbifold construction and performed a number of simple checks on the resulting system. It should not be complicated to compare the Bethe ansatz predictions with the classical sigma model on $AdS_5 \times S^5/\Gamma$. For that one would have to implement the orbifold projection at the level of the spectral curve of the sigma model on $AdS_5 \times S^5$.

Our discussion focused on abelian orbifolds. From a field theory standpoint it is possible to construct non-abelian orbifold projections as well. It would be interesting to understand the consequences of such a projection on the integrable structure of the parent theory. For such theories the twisted sectors are associated to the conjugacy classes of the orbifold group and, consequently, the analog of (3.11) becomes more involved. Moreover, it is likely to be necessary to include more than one quasi-excitation which also exhibit non-trivial scattering phases among themselves. It might also be worth considering more general theories with $\mathcal{N} < 4$ supersymmetry [42 − 45] such as quiver gauge theories [46] and model with fundamental matter [37 − 41].

It would also be interesting to understand the details of the gauge and string theory description of orbifolds of AdS_5. As we have already mentioned, the from the standpoint of

[5] Some integrable open spin chains with more general boundary conditions are also connected to the one-loop dilatation operators for theories with fields in the fundamental representation [37 − 41].

the Bethe ansatz it merely requires allowing the non-trivial orbifold weights **s** corresponding to the roots of the conformal group, the details of the field theory leading to such an ansatz (even at one loop) are not completely clear.

A very interesting problem is understanding the effects on the anomalous dimensions of the double-trace deformations induced at loop level [19, 20, 22] in the absence of supersymmetry. As we have already mentioned, for length L operators these deformations cannot have an effect of order one before the $L-1$ loop order. From this perspective their effects are not unlike those of the wrapping interactions and understanding how to incorporate the former in the Bethe ansatz may shed light on the latter.

Acknowledgments

We would like to thank Didina Serban and Mathias Staudacher for discussions. R.R. is grateful for hospitality at KITP and Princeton University where parts of this work was carried out. R.R. acknowledges partial support of NSF grant PHY99-07949 while at KITP. The work of N.B. is supported in part by the U.S. National Science Foundation Grant No. PHY02-43680. Any opinions, findings and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

References

[1] J.A. Minahan and K. Zarembo, *The Bethe-ansatz for $N=4$ super Yang-Mills*, *JHEP* **03** (2003) 013 [hep-th/0212208].

[2] N. Beisert, C. Kristjansen and M. Staudacher, *The dilatation operator of $N=4$ super Yang-Mills theory*, *Nucl. Phys.* **B 664** (2003) 131 [hep-th/0303060].

[3] N. Beisert and M. Staudacher, *The $N=4$ SYM integrable super spin chain*, *Nucl. Phys.* **B 670** (2003) 439 [hep-th/0307042].

[4] N. Beisert, *The* su(2|3) *dynamic spin chain*, *Nucl. Phys.* **B 682** (2004) 487 [hep-th/0310252].

[5] G. Mandal, N.V. Suryanarayana and S.R. Wadia, *Aspects of semiclassical strings in AdS_5*, *Phys. Lett.* **B 543** (2002) 81 [hep-th/0206103].

[6] I. Bena, J. Polchinski and R. Roiban, *Hidden symmetries of the $AdS_5 \times S^5$ superstring*, *Phys. Rev.* **D 69** (2004) 046002 [hep-th/0305116].

[7] V.A. Kazakov, A. Marshakov, J.A. Minahan and K. Zarembo, *Classical/quantum integrability in AdS/CFT*, *JHEP* **05** (2004) 024 [hep-th/0402207].

[8] N. Beisert, *The dilatation operator of $N=4$ super Yang-Mills theory and integrability*, *Phys. Rept.* **405** (2005) 1 [hep-th/0407277].

[9] N. Beisert, *Higher-loop integrability in $N=4$ gauge theory*, *Comptes Rendus Physique* **5** (2004) 1039–1048 [hep-th/0409147].

[10] K. Zarembo, *Semiclassical Bethe ansatz and AdS/CFT*, *Comptes Rendus Physique* **5** (2004) 1081–1090 [hep-th/0411191].

[11] J. Plefka, *Spinning strings and integrable spin chains in the AdS/CFT correspondence*, hep-th/0507136.

[12] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[13] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109].

[14] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[15] S. Kachru and E. Silverstein, *4D conformal theories and strings on orbifolds*, *Phys. Rev. Lett.* **80** (1998) 4855 [hep-th/9802183].

[16] A.E. Lawrence, N. Nekrasov and C. Vafa, *On conformal field theories in four dimensions*, *Nucl. Phys.* **B 533** (1998) 199 [hep-th/9803015].

[17] M. Bershadsky, Z. Kakushadze and C. Vafa, *String expansion as large-N expansion of gauge theories*, *Nucl. Phys.* **B 523** (1998) 59 [hep-th/9803076].

[18] M. Bershadsky and A. Johansen, *Large-N limit of orbifold field theories*, *Nucl. Phys.* **B 536** (1998) 141 [hep-th/9803249].

[19] A.A. Tseytlin and K. Zarembo, *Effective potential in non-supersymmetric* $\mathrm{SU}(N) \times \mathrm{SU}(N)$ *gauge theory and interactions of type 0 D3-branes*, *Phys. Lett.* **B 457** (1999) 77 [hep-th/9902095].

[20] A. Dymarsky, I.R. Klebanov and R. Roiban, *Perturbative search for fixed lines in large-N gauge theories*, *JHEP* **08** (2005) 011 [hep-th/0505099].

[21] A. Adams and E. Silverstein, *Closed string tachyons, AdS/CFT and large-N QCD*, *Phys. Rev.* **D 64** (2001) 086001 [hep-th/0103220].

[22] A. Dymarsky, I.R. Klebanov and R. Roiban, *Perturbative gauge theory and closed string tachyons*, hep-th/0509132.

[23] M. Alishahiha and M.M. Sheikh-Jabbari, *The pp-wave limits of orbifolded* $AdS_5 \times S^5$, *Phys. Lett.* **B 535** (2002) 328 [hep-th/0203018].

[24] N.W. Kim, A. Pankiewicz, S.J. Rey and S. Theisen, *Superstring on pp-wave orbifold from large-N quiver gauge theory*, *Eur. Phys. J.* **C 25** (2002) 327 [hep-th/0203080].

[25] X.-J. Wang and Y.-S. Wu, *Integrable spin chain and operator mixing in* $N = 1,2$ *supersymmetric theories*, *Nucl. Phys.* **B 683** (2004) 363 [hep-th/0311073].

[26] K. Ideguchi, *Semiclassical strings on* $AdS_5 \times S^5/\mathbb{Z}(m)$ *and operators in orbifold field theories*, *JHEP* **09** (2004) 008 [hep-th/0408014].

[27] N. Beisert, V. Dippel and M. Staudacher, *A novel long range spin chain and planar* $N = 4$ *super Yang-Mills*, *JHEP* **07** (2004) 075 [hep-th/0405001].

[28] J. Ambjørn, R.A. Janik and C. Kristjansen, *Wrapping interactions and a new source of corrections to the spin-chain/string duality*, hep-th/0510171.

[29] N. Beisert and M. Staudacher, *Long-range* $\mathrm{PSU}(2,2|4)$ *Bethe ansatz for gauge theory and strings*, *Nucl. Phys.* **B 727** (2005) 1 [hep-th/0504190].

[30] G. Arutyunov, S. Frolov and M. Staudacher, *Bethe ansatz for quantum strings*, *JHEP* **10** (2004) 016 [hep-th/0406256].

[31] R. Roiban, *On spin chains and field theories*, *JHEP* **09** (2004) 023 [hep-th/0312218].

[32] D. Berenstein and S.A. Cherkis, *Deformations of $N = 4$ SYM and integrable spin chain models*, *Nucl. Phys.* **B 702** (2004) 49 [hep-th/0405215].

[33] O. Lunin and J. Maldacena, *Deforming field theories with* $\mathrm{U}(1) \times \mathrm{U}(1)$ *global symmetry and their gravity duals*, *JHEP* **05** (2005) 033 [hep-th/0502086].

[34] S.A. Frolov, R. Roiban and A.A. Tseytlin, *Gauge-string duality for superconformal deformations of $N = 4$ super Yang-Mills theory*, *JHEP* **07** (2005) 045 [hep-th/0503192].

[35] S. Frolov, *Lax pair for strings in Lunin-Maldacena background*, *JHEP* **05** (2005) 069 [hep-th/0503201].

[36] N. Beisert and R. Roiban, *Beauty and the twist: the Bethe ansatz for twisted $N = 4$ SYM*, *JHEP* **08** (2005) 039 [hep-th/0505187].

[37] B. Chen, X.J. Wang and Y.S. Wu, *Integrable open spin chain in super Yang-Mills and the plane-wave/SYM duality*, *JHEP* **02** (2004) 029 [hep-th/0401016].

[38] O. DeWolfe and N. Mann, *Integrable open spin chains in defect conformal field theory*, *JHEP* **04** (2004) 035 [hep-th/0401041].

[39] T.G. Erler and N. Mann, *Integrable open spin chains and the doubling trick in $N = 2$ SYM with fundamental matter*, hep-th/0508064.

[40] D. Berenstein and S.E. Vazquez, *Integrable open spin chains from giant gravitons*, *JHEP* **06** (2005) 059 [hep-th/0501078].

[41] T. McLoughlin and I.J. Swanson, *Open string integrability and AdS/CFT*, *Nucl. Phys.* **B 723** (2005) 132 [hep-th/0504203].

[42] G. Ferretti, R. Heise and K. Zarembo, *New integrable structures in large-N QCD*, *Phys. Rev.* **D 70** (2004) 074024 [hep-th/0404187].

[43] P. Di Vecchia and A. Tanzini, *$N = 2$ super Yang-Mills and the XXZ spin chain*, *J. Geom. Phys.* **54** (2005) 116 [hep-th/0405262].

[44] A.V. Belitsky, V.M. Braun, A.S. Gorsky and G.P. Korchemsky, *Integrability in QCD and beyond*, *Int. J. Mod. Phys.* **A 19** (2004) 4715 [hep-th/0407232].

[45] N. Beisert, G. Ferretti, R. Heise and K. Zarembo, *One-loop QCD spin chain and its spectrum*, *Nucl. Phys.* **B 717** (2005) 137 [hep-th/0412029].

[46] M.R. Douglas and G.W. Moore, *D-branes, quivers and ALE instantons*, hep-th/9603167.

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